The Seiberg–Witten map for the 4D noncommutative BF theory

L C Q Vilar[1], **O S Ventura**[2,3], **R L P G Amaral**[4], **V E R Lemes**[1]
and L O Buffon[3,5]

[1] Instituto de Física, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, Maracanã, Rio de Janeiro - RJ, 20550-013, Brazil
[2] Coordenadoria de Física, Centro Federal de Educação Tecnológica do Espírito Santo, Avenida Vitória 1729 - Jucutuquara, Vitória - ES, 29040 - 333, Brazil
[3] Centro Universitário de Vila Velha, Rua comissário José Dantas de Mello 15 - Boa Vista, Vila Velha - ES, 29102 - 770, Brazil
[4] Instituto de Física, Universidade Federal Fluminense, 24210 - 340, Niterói - RJ, Brasil
[5] Escola Superior de Ciências da Santa Casa de Misericórdia de Vitória, Av. Nossa Senhora da Penha 2190, Santa Luiza, Vitória-ES, 29045-402, Brazil

E-mail: mbschlee@terra.com.br, ozemar@cefetes.br, rubens@if.uff.br, vitor@dft.if.uerj.br and lobuffon@terra.com.br

Abstract

We describe the Seiberg–Witten map taking the 4D noncommutative BF theory (NCBF) into its pure commutative version. The existence of this map is in agreement with the hypothesis that such maps are available for any noncommutative theory with Schwarz-type topological sectors, and represents a strong indication for the renormalizability of these theories in general.

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1. Introduction

Distinct gauge choices in the open strings lead to the realization of both ordinary Yang–Mills field theories as well as noncommutative field theories. It was the perception of this fact that made Seiberg and Witten propose the so-called Seiberg–Witten map (SW map) [1]. This mapping establishes an expression of noncommutative field variables in terms of ordinary (commutative) fields, in such a way that the noncommutative gauge transformed fields are mapped into commutative fields gauge transformed in the ordinary sense.

Let us briefly review this idea. First, we introduce the Moyal product between two functions defined on the noncommutative space [2],

$$
\begin{aligned}
f * g &= \exp\left(\frac{\mathrm{i}}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)g(y)_{y\to x} \\
&= fg + \frac{\mathrm{i}}{2}\theta^{ij}\partial_i f \partial_j g + O(\theta^2),
\end{aligned}
\tag{1}
$$

where the real c-number parameter θ_{ij} is originated in the noncommutativity of spacetime coordinates

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\theta_{ij}. \tag{2}$$

The vanishing of θ turns the noncommutative theory into the commutative one. Then, after defining the Moyal bracket,

$$[f \overset{*}{,} g] = f * g - g * f, \tag{3}$$

the noncommutative gauge transformation is constructed as

$$\hat{\delta}_{\hat{\lambda}} \hat{A} = \partial_i \hat{\lambda} + \mathrm{i}[\hat{\lambda} \overset{*}{,} \hat{A}_i] = \hat{D}_i \hat{\lambda} . \tag{4}$$

Here the hat symbol identifies fields and operators defined on the noncommutative spacetime while $\hat{D}_i$ represents the Moyal covariant derivative. At this point it should be observed that, although similar in form to a non-Abelian gauge transformation, there is a nonvanishing contribution coming from the Moyal bracket in (4) even for an Abelian gauge field. In the same way, the noncommutative gauge curvature

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - \mathrm{i}[\hat{A}_i \overset{*}{,} \hat{A}_j] \tag{5}$$

presents a nonvanishing commutator contribution even for an Abelian field.

In the Abelian case the above expressions, up to first order in θ, turn out to be

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_i = \partial_i \hat{\lambda} - \theta^{kl} \partial_k \hat{\lambda} \partial_l \hat{A}_i + O(\theta^2), \tag{6}$$

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \theta^{kl} \partial_k \hat{A}_i \partial_l \hat{A}_j + O(\theta^2). \tag{7}$$

Now, the sense of the SW map is that noncommutative gauge equivalent fields should be mapped into ordinary gauge equivalent fields. A particular solution for this problem in the $U(1)$ case, again up to first order in θ, is given by [1]

$$\hat{A}_i(A) = A_i - \theta^{kl}\left(A_k \partial_l A_i - \tfrac{1}{2} A_k \partial_i A_l\right) + O(\theta^2), \tag{8}$$

$$\hat{\lambda}(\lambda, A) = \lambda + \tfrac{1}{2}\theta^{kl}(\partial_k \lambda) A_l + O(\theta^2), \tag{9}$$

and a cohomological characterization of the general solution was accomplished in [3]. As a by-product of this study, a conjecture on the existence of SW maps taking noncommutative theories in the presence of topological terms into renormalizable commutative field theories was proposed. The case of 3D theories was analysed in detail. It was shown how a map of the 3D noncommutative Maxwell theory in the presence of a noncommutative Chern–Simons term (NCCSM) would lead to the usual commutative theory [3], whereas θ-dependent nonrenormalizable interactions inevitably accompany the SW mapped noncommutative Maxwell theory (NCM) theory in the absence of a topological sector [4].

In fact, the lack of renormalizability of noncommutative gauge theories already appeared at the noncommutative space, when it was noted that infrared and ultraviolet divergencies were deeply entangled [5, 6]. So it could not be a surprise at all to find that the SW mapping of such theories led to commutative theories presenting those θ-dependent nonrenormalizable couplings. Since then, some evolution has been made in this topic. Grosse and Wulkenhaar proposed to couple oscillator-like terms to the original noncommutative theories in order to avoid the IR/UV mixing [7]. Slavnov also proposed the coupling of a new term to render the

gauge action renormalizable [8]. Recently, it was shown that this new term was effectively a BF-like coupling in the noncommutative space, and that the renormalizability so achieved should be expected due to the strength of the supersymmetry present in topological theories [9]. This is in agreement with the idea that a SW map can always join noncommutative theories with topological sectors together with the renormalizable commutative theories. But, about this point, some evidences pointing towards different conclusions can be found in the literature. Blasi and Maggiori argued for quantum instabilities in 2D NCBF theory and conjectured that this would be a general feature of noncommutative theories in higher dimensions [10], although in 3D they agreed that the 'instabilities' of noncommutative Chern–Simons (NCCS) would be harmless as they were always restricted to the trivial BRST sector [11], thus supporting again the hypothesis that the renormalizability is connected to the existence of a SW mapped renormalizable commutative version of the original noncommutative theory, as it happens with NCCS [12]. These theories, as NCCS, even having non-power-counting couplings in the noncommutative space, would be renormalizable in a broader sense [13, 14].

So, it is an open question whether in 4D the NCBF theory is renormalizable and if there is a SW map taking this theory into a renormalizable commutative one. In view of the breakdown of the renormalizability of 4D noncommutative Yang–Mills (NCYM) theories (and also remembering that commutative YM theory has already been consistently proposed to be seen as a deformation of a topological BF theory [15]) answering these questions become relevant to the fate of noncommutative 4D theories. Here we want to give a definitive answer to the second question, showing the existence of the SW map for NCBF theory into commutative pure BF theory without any nonrenormalizable coupling, in much the same way as it happens in the 3D NCCS case [3, 12].

In the following section, we will show that the noncommutative space allows the existence of new symmetries not reducible to commutative ones. These symmetries are necessary to construct the SW map of the NCBF theory, and we give the example of this map in the first order in θ. In section 3 we generalize the reasoning of the first order by means of a BRST technique, proving the existence of the SW map of the NCBF theory to all orders. In the conclusion, we show that this map takes the NCBF action into the pure commutative BF action, without the presence of any nonrenormalizable couplings. This supports the conjecture on the renormalizability of the SW mapped theories coming from noncommutative theories with topological sectors in 4D.

2. The first-order SW map

Let us begin by writing the action for the NCBF theory

$$\mathcal{S}_{\widehat{BF}} = \frac{1}{4} \int \mathrm{d}^4 x \, \varepsilon^{\mu\nu\varrho\sigma} \, \hat{B}_{\mu\nu} * \hat{F}_{\varrho\sigma}. \tag{10}$$

The map that we will search, $\hat{A} = \hat{A}(A, B)$ and $\hat{B} = \hat{B}(A, B)$, should now preserve the gauge symmetry

$$\overset{\wedge}{\delta}_{\overset{\wedge}{\lambda}} \hat{A}_\mu(A, B) = \delta_\lambda(\hat{A}_\mu(A, B)), \qquad \overset{\wedge}{\delta}_{\overset{\wedge}{\lambda}} \hat{B}_{\mu\nu}(A, B) = \delta_\lambda(\hat{B}_{\mu\nu}(A, B)), \tag{11}$$

as well as the topological symmetry typical of BF theories

$$\overset{\wedge}{\delta}_{\overset{\wedge}{\psi}} \hat{A}_\mu(A, B) = \delta_\psi(\hat{A}_\mu(A, B)), \qquad \overset{\wedge}{\delta}_{\overset{\wedge}{\psi}} \hat{B}_{\mu\nu}(A, B) = \delta_\psi(\hat{B}_{\mu\nu}(A, B)), \tag{12}$$

for both fields simultaneously, where, in the Abelian case,

$$\overset{\wedge}{\delta}_{\overset{\wedge}{\lambda}} \hat{A}_\mu = \hat{D}_\mu \overset{\wedge}{\lambda}, \, \overset{\wedge}{\delta}_{\overset{\wedge}{\lambda}} \hat{B}_{\mu\nu} = \mathrm{i}[\overset{\wedge}{\lambda} \overset{*}{,} \hat{B}_{\mu\nu}], \tag{13}$$

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$$\delta_\lambda A_\mu = \partial_\mu \lambda, \qquad \delta_\lambda B_{\mu\nu} = 0, \tag{14}$$

and

$$\hat{\delta}_{\hat{\psi}} \hat{A}_\mu = 0, \qquad \hat{\delta}_{\hat{\psi}} \hat{B}_{\mu\nu} = \hat{D}_\mu \hat{\psi}_\nu - \hat{D}_\nu \hat{\psi}_\mu, \tag{15}$$

$$\hat{D}_\mu \hat{\psi}_\nu = \partial_\mu \hat{\psi}_\nu + \mathrm{i}[\hat{\psi}_\nu \overset{*}{,} \hat{A}_\mu]. \tag{16}$$

$$\delta_\psi A_\mu = 0, \qquad \delta_\psi B_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu. \tag{17}$$

The problem that emerges when we try to construct the Seiberg–Witten map of the noncommutative BF model is the impossibility of the simultaneous implementation of the conditions shown in (11) and (12). The solution of the SW condition for the $\hat{A}_\mu$ field is actually straightforward since $\hat{\delta}_{\hat{\psi}} \hat{A}_\mu = 0$. The SW mapping for this sector reduces to the SW problem of purely vectorial field theories. The cohomological treatment for this specific case was presented by us in [3]. Then, the main difficulty is in the search for solutions of the $\hat{B}_{\mu\nu}$ field mapping, and, of course, this difficulty is related to the topological symmetry. (Indeed a treatment that does not take into account the topological symmetry was presented in [16]. There, the addition to (10) of a quadratic term of the type $\hat{B}_{\mu\nu} \hat{B}^{\mu\nu}$ explicitly broke the topological symmetry, turning it possible the implementation of the usual SW mapping.)

So, we start our search by noting that condition (12), although natural, is not the most general one. The presence of a noncommutative structure allows an infinite set of new symmetries that can be used to extend the $\hat{\delta}_{\hat{\psi}} \hat{B}$ symmetry and then deform (12). The extended symmetry remains an invariance of the noncommutative action (10).

For example, in the first explicit order in θ, the following operation:

$$\hat{\delta}_{\hat{T}} \hat{B}_{\mu\nu} = -\tfrac{1}{2}\theta^{\alpha\beta}\{\hat{D}_\alpha \hat{F}_{\mu\nu} \overset{*}{,} \hat{\Psi}_\beta\} + \tfrac{1}{4}\theta^{\alpha\beta}(\{\hat{D}_\beta \hat{\Psi}_{[\nu} \overset{*}{,} \hat{F}_{\mu]\alpha}\} - \{\hat{D}_{[\nu}\hat{\Psi}_\beta \overset{*}{,} \hat{F}_{\mu]\alpha}\}), \qquad \hat{\delta}_{\hat{T}} \hat{A}_\mu = 0, \tag{18}$$

generates a symmetry of (10),

$$\hat{\delta}_{\hat{T}} \mathcal{S}_{\widehat{BF}} = 0. \tag{19}$$

It is important to remark that this new symmetry (18) was picked up among the set of possible symmetries with one θ because it has a parameter $\hat{\Psi}$ which fits the same dimensionality of the parameter $\hat{\psi}$ of the topological symmetry (15). There are other BRST equivalent ways of writing the symmetry in (18) with the parameter $\hat{\Psi}$, as we will see after equation (41). In this sense, the choice of (18) is unique. There are also other possible symmetries of (10), but with parameters of different dimensions which are not suitable for our problem. As it will be seen, (18) is the 'viable' deformation of (12) that we were looking for. Then, by identifying both parameters, we propose that, at least in first order in θ, the SW condition should relate this extended noncommutative topological symmetry

$$\hat{\delta}_{\hat{\Psi}} \hat{B}_{\mu\nu} = \hat{\delta}_{\hat{\psi}} \hat{B}_{\mu\nu} + \hat{\delta}_{\hat{T}} \hat{B}_{\mu\nu}$$

$$= \hat{D}_{[\mu} \hat{\Psi}_{\nu]} - \tfrac{1}{2}\theta^{\alpha\beta}\{\hat{D}_\alpha \hat{F}_{\mu\nu} \overset{*}{,} \hat{\Psi}_\beta\} + \tfrac{1}{4}\theta^{\alpha\beta}(\{\hat{D}_\beta \hat{\Psi}_{[\nu} \overset{*}{,} \hat{F}_{\mu]\alpha}\} - \{\hat{D}_{[\nu}\hat{\Psi}_\beta \overset{*}{,} \hat{F}_{\mu]\alpha}\}) \tag{20}$$

and the standard commutative topological symmetry of (17), i.e.,

$$\hat{\delta}_{\hat{\Psi}} \hat{B}_{\mu\nu}(A, B) = \delta_\psi (\hat{B}_{\mu\nu}(A, B)). \tag{21}$$

Now, this condition has a solution at first order in θ that can be presented as

$$\hat{B}{}^{1}_{\mu\nu} = -\theta^{\alpha\beta} A_\alpha \partial_\beta B_{\mu\nu} - \tfrac{1}{2}\theta^{\alpha\beta} F_{[\mu\alpha} B_{\beta\nu]} \tag{22}$$

$$\overset{\wedge}{\Psi}{}^{1}_{\mu} = -\theta^{\alpha\beta} A_\alpha \partial_\beta \psi_\mu + \theta^{\alpha\beta} F_{\alpha\mu} \psi_\beta \tag{23}$$

But it is not difficult to see that, at second order in θ, the new condition (21) has again an obstruction. Using the same reasoning, one can search for another symmetry of the $\widehat{BF}$ theory with a parameter with the same dimensions as $\overset{\wedge}{\psi}$, now at the (explicit) second order in θ; then one can extend the symmetry (20) once more, and finally construct a new condition substituting (21), which will be solvable at second order. This can be achieved, but it is not surprising at all to understand that in the end the obstruction in the solution of the SW map for the topological symmetry will move to the next order and so on.

In this scenario, the sensible question is if it is possible to assure the existence of an extended noncommutative topological symmetry at all orders, from which one could extract the definite SW map of the noncommutative $\widehat{BF}$ theory, and then find its form in the commutative space.

In the following we will prove the existence of such map.

3. The existence of the SW map at all orders

The starting point is the BRST transformations of the noncommutative fields and ghosts. Analogously to the non-Abelian commutative BF case, that is fully discussed in [17, 18], they are given by

$$\begin{aligned}
\hat{s}\hat{A}_\mu &= \hat{D}_\mu \hat{c}, \\
\hat{s}\hat{c} &= \mathrm{i}\,\hat{c} * \hat{c}, \\
\hat{s}\hat{B}_{\mu\nu} &= \hat{D}_{[\mu}\overset{\wedge}{\psi}_{\nu]} + \mathrm{i}[\hat{c}\overset{*}{,}\hat{B}_{\mu\nu}], \\
\hat{s}\overset{\wedge}{\psi}_\mu &= \hat{D}_\mu \overset{\wedge}{\rho} + \mathrm{i}[\hat{c}\overset{*}{,}\overset{\wedge}{\psi}_\mu], \\
\hat{s}\overset{\wedge}{\rho} &= \mathrm{i}[\hat{c}\overset{*}{,}\overset{\wedge}{\rho}].
\end{aligned} \tag{24}$$

Let us mention that the ghost $\widehat{\rho}$ is necessary due to the zero modes in the transformation of $\overset{\wedge}{\psi}_\mu$.

In this BRST context, the SW map is the solution of the condition relating the noncommutative BRST transformations and those of the commutative theory. For example, the conditions for $\hat{A}_\mu$ and $\hat{c}$,

$$\hat{s}\hat{A}_\mu(A) = s(\hat{A}_\mu(A)), \tag{25}$$

$$\hat{s}\hat{c}(A, c) = s(\hat{c}(A, c)), \tag{26}$$

have the same structure of the pure noncommutative Maxwell case. Then, we can expect the same solution as we anticipated in (8) and (9). As we are searching for an argument valid for all orders, let us derive the relations expressing the full dependence of the mapped fields

on θ. The commutative operator s obviously does not depend on θ, and we can establish the commutation between s and the operator of the variation under θ, δ_θ,

$$[s, \delta_\theta] = 0. \tag{27}$$

Now, from this relation, the SW conditions (25), (26), and the BRST transformations (24), we see that the SW mapped fields $\hat{c}$ and $\hat{A}_\mu$ should satisfy

$$\hat{s}(\delta_\theta \hat{c}) = i\{\delta_\theta \hat{c}^* \hat{c}\} - \tfrac{1}{4}\delta\theta^{\alpha\beta}[\partial_\alpha \hat{c}^* \partial_\beta \hat{c}], \tag{28}$$

$$\hat{s}(\delta_\theta \hat{A}_\mu) = \hat{D}_\mu(\delta_\theta \hat{c}) + i[\delta_\theta \hat{c}^* \hat{A}_\mu] + i[\hat{c}^* \delta_\theta \hat{A}_\mu] - \tfrac{1}{2}\delta\theta^{\alpha\beta}[\partial_\alpha \hat{c}^* \partial_\beta \hat{A}_\mu]. \tag{29}$$

A particular solution to this system is given by

$$\delta_\theta \hat{c} = \tfrac{1}{4}\delta\theta^{\alpha\beta}\{\partial_\alpha \hat{c}^* \hat{A}_\beta\}, \tag{30}$$

$$\delta_\theta \hat{A}_\mu = -\tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{A}_\mu^* \partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu}\}. \tag{31}$$

These solutions were written for the first time in [1], although they were derived there far from the BRST environment. As stressed above, here they just represent particular solutions to the system (28), (29). In fact, once the nilpotency of the BRST operator $\hat{s}$ is assured, the complete solution of the problem requires a characterization of the cohomological classes of $\hat{s}$ with the convenient quantum numbers. A general analysis (which is not of our concern here) should follow the lines of the work in [3]. For the moment, we would like to call attention to the fact that solutions (30) and (31) are defined modulo covariant elements under $\hat{s}$, i.e., objects that transform as

$$\hat{s}\hat{X} = i[\hat{c}^* \hat{X}]. \tag{32}$$

Note also that equations (8) and (9) can be seen as first-order solutions of (30) and (31). It is worthwhile to reinforce that the obtention of the differential equations (30) and (31) is sufficient to assure the existence of the SW map order by order in θ, as explained in [1].

This procedure can now be followed for the other fields and ghosts of the $\widehat{BF}$ theory appearing in (24). The solutions for the θ variations of the ghosts $\hat{\rho}$ and $\hat{\psi}_\mu$ are obtained in an analogous and straightforward way and are given by

$$\delta_\theta \hat{\rho} = -\tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{A}_\alpha^* (\partial_\beta + \hat{D}_\beta)\hat{\rho}\}, \tag{33}$$

$$\delta_\theta \hat{\psi}_\mu = -\tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{A}_\alpha^*(\partial_\beta + \hat{D}_\beta) \hat{\psi}_\mu\} + \tfrac{1}{2}\delta\theta^{\alpha\beta}\{\hat{F}_{\alpha\mu}^* \hat{\psi}_\beta\}. \tag{34}$$

Here we must call attention once more for the importance of obtaining relations (30), (31), (33) and (34). First of all, we stress that these are exact expressions for the differentials $\delta_\theta \hat{A}_\mu$, $\delta_\theta \hat{c}$, $\delta_\theta \hat{\rho}$ and $\delta_\theta \hat{\psi}_\mu$ valid to all orders in the θ expansion. From them, we can derive the SW map order by order in θ, by reintroducing the lowest order solutions in the equations and then solving for the next higher order. This procedure is a systematic method for the study of the existence of the SW map for a general set of transformations, allowing then the obtention of the explicit SW maps of the fields to any order in $\theta^{\mu\nu}$. In other words, if we are able to show such differentials for all the fields on the noncommutative theory, then the existence of the SW map for the whole theory is assured. So, we must find the same differential for the last remaining field, $\hat{B}_{\mu\nu}$.

Nevertheless the situation of the $\hat{B}_{\mu\nu}$ field is more involved, and we will analyse it in detail. The problem that we will describe is in fact the root of all the difficulties that we have been meeting since the beginning of our work. As we just mentioned, the nilpotency of the BRST operator $\hat{s}$ is the fundamental pillar of all this construction. This can be understood by remembering that the SW condition on $\hat{B}_{\mu\nu}$ leads to

$$\hat{s}^2 \hat{B}_{\mu\nu}(A, B) = s^2(\hat{B}_{\mu\nu}(A, B)), \tag{35}$$

and as the BRST operator s is nilpotent on all fields of the Abelian BF theory, the existence of the SW map becomes conditioned to the nilpotency of $\hat{s}$ as well. But the resemblance of the BRST transformations (24) on the set of transformations of the commutative non-Abelian BF case brings in a well-known problem of the later: the lack of nilpotency of the BRST operator for the non-Abelian BF system. This only comes into play now because it happens precisely on the $\hat{B}_{\mu\nu}$ field,

$$\hat{s}^2 \hat{B}_{\mu\nu} = -\mathrm{i}[\hat{F}_{\mu\nu} \overset{*}{,} \overset{\wedge}{\rho}]. \tag{36}$$

In the quantum treatment of the commutative non-Abelian case this is overcome using the Batalin–Vilkovisky procedure by introducing a term of higher order in the anti-fields in the fully quantized action [18, 19]. Or, in the BRST language, noting that we are dealing with a topological field theory, we have at our disposal a complete ladder structure [20] which makes it immediate the correction of the BRST transformation of $\hat{B}_{\mu\nu}$ in order to build a nilpotent Slavnov operator [18]. In our case we have an alternative allowed by the presence of the noncommutative θ parameter. Let us deform the transformation of the $\widehat{B}_{\mu\nu}$ field in (24) as follows:

$$\hat{s}\hat{B}_{\mu\nu} = \hat{D}_{[\mu}\overset{\wedge}{\psi}_{\nu]} + \mathrm{i}[\hat{c} \overset{*}{,} \hat{B}_{\mu\nu}] + \overset{\wedge}{\delta}_t \hat{B}_{\mu\nu}, \tag{37}$$

in such a way that the nilpotency of $\hat{s}$ can be recovered,

$$\hat{s}^2 \hat{B}_{\mu\nu} = -\mathrm{i}[\hat{F}_{\mu\nu} \overset{*}{,} \overset{\wedge}{\rho}] + \hat{s}(\overset{\wedge}{\delta}_t \hat{B}_{\mu\nu}) - \mathrm{i}[\hat{c} \overset{*}{,} \overset{\wedge}{\delta}_t \hat{B}_{\mu\nu}] = 0. \tag{38}$$

At this point, we can see that we are generalizing the procedure that was taken in the introduction of this work when we were dealing with the θ first-order case (equation (37) is the all orders generalization of (20)).

A brief comment is important here. In the BV language, in the presence of the anti-fields, the idea of deforming the set of field transformations of a given theory together with the deformation of its action is well known in the literature [21]. In fact, equation (38) when translated to the BV language, would appear as the correction demanded by the master equation (it would come from the term joining the transformation of the ghost $\overset{\wedge}{\psi}_\mu$ an of its anti-fields). This correction would just be the introduction of the quadratic term in the anti-field of $\hat{B}_{\mu\nu}$ in the BV action, as it happens in the commutative case that we have. But as we are seeking for a SW map transforming the fields of the noncommutative $\hat{B}F$ theory into those of the usual commutative theory, we avoid the introduction of the BV anti-fields. This is only possible due to the existence of the noncommutative parameter $\theta^{\mu\nu}$, which allows us to restrain ourselves to the set of the fields of the theory without the presence of the anti-fields. Then we find this present approach a more direct attack to the SW map problem.

The solution of (38) is thus the key ingredient to assert the existence of the SW mapping, implying that

$$\hat{s}(\overset{\wedge}{\delta}_t \hat{B}_{\mu\nu}) = \mathrm{i}[\hat{F}_{\mu\nu} \overset{*}{,} \overset{\wedge}{\rho}] + \mathrm{i}[\hat{c} \overset{*}{,} \overset{\wedge}{\delta}_t \hat{B}_{\mu\nu}]. \tag{39}$$

We can now act with δ_θ on this equation,

$$\hat{s}(\delta_\theta \hat{\delta}_t \hat{B}_{\mu\nu}) = \mathrm{i}[\delta_\theta \hat{F}_{\mu\nu} \overset{*}{,} \hat{\rho}] - \tfrac{1}{2}\delta\theta^{\alpha\beta}\{\partial_\alpha \hat{F}_{\mu\nu} \overset{*}{,} \partial_\beta \hat{\rho}\} + \mathrm{i}[\hat{F}_{\mu\nu} \overset{*}{,} \delta_\theta \hat{\rho}]$$
$$+ \mathrm{i}[\delta_\theta \hat{c} \overset{*}{,} \hat{\delta}_t \hat{B}_{\mu\nu}] - \tfrac{1}{2}\delta\theta^{\alpha\beta}\{\partial_\alpha \hat{c} \overset{*}{,} \partial_\beta(\hat{\delta}_t \hat{B}_{\mu\nu})\} + \mathrm{i}[\hat{c} \overset{*}{,} \delta_\theta \hat{\delta}_t \hat{B}_{\mu\nu}], \tag{40}$$

use (30), (31), (33), and, then, find

$$\delta_\theta \hat{\delta}_t \hat{B}_{\mu\nu} = -\tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{A}_\alpha \overset{*}{,} (\partial_\beta + \hat{D}_\beta)\hat{\delta}_t \hat{B}_{\mu\nu}\} - \tfrac{1}{2}\delta\theta^{\alpha\beta}\{\hat{D}_\alpha \hat{F}_{\mu\nu} \overset{*}{,} \hat{\psi}_\beta\}$$
$$- \tfrac{1}{2}\delta\theta^{\alpha\beta}\{\hat{F}_{[\mu\alpha} \overset{*}{,} \hat{\delta}_t \hat{B}_{\beta\nu]}\} + \delta\theta^{\alpha\beta}\hat{X}_{\mu\nu\alpha\beta}, \tag{41}$$

where $\hat{X}_{\mu\nu\alpha\beta}$ represents the freedom in the solution of this kind of problem by covariant terms, as we saw in (32),

$$\hat{s}\hat{X}_{\mu\nu\alpha\beta} = \mathrm{i}[\hat{c} \overset{*}{,} \hat{X}_{\mu\nu\alpha\beta}]. \tag{42}$$

But there are still some restrictions on solution (41). When we considered the deformation of the transformation of $\hat{B}_{\mu\nu}$ in (37), we obviously intended that it would still represent a symmetry of the action. This implies that (taking $\hat{\delta}_t \hat{A}_\mu = 0$)

$$\hat{\delta}_t \mathcal{S}_{\widehat{BF}} = \int \mathrm{d}^4x \, \frac{1}{4}\varepsilon^{\mu\nu\varrho\sigma} \hat{F}_{\mu\nu} \hat{\delta}_t \hat{B}_{\varrho\sigma} = 0 \tag{43}$$

and

$$\delta_\theta \hat{\delta}_t \mathcal{S}_{\widehat{BF}} = \int \mathrm{d}^4x \, \frac{1}{4}\varepsilon^{\mu\nu\varrho\sigma}(\delta_\theta \hat{F}_{\mu\nu} \hat{\delta}_t \hat{B}_{\varrho\sigma} + \hat{F}_{\mu\nu} \, \delta_\theta \hat{\delta}_t \hat{B}_{\varrho\sigma}) = 0. \tag{44}$$

Substituting (41) into (44) we get

$$\int \mathrm{d}^4x \, \varepsilon^{\mu\nu\rho\sigma} \delta\theta^{\alpha\beta}\left(\frac{1}{2}\{\hat{F}_{\mu\nu} \overset{*}{,} \hat{F}_{\alpha\rho}\}\hat{\delta}_t \hat{B}_{\beta\sigma} - \frac{1}{2}\hat{F}_{\mu\nu}\{\hat{D}_\alpha \hat{F}_{\rho\sigma} \overset{*}{,} \hat{\psi}_\beta\} + \hat{F}_{\rho\sigma}\hat{X}_{\mu\nu\alpha\beta}\right) = 0 \tag{45}$$

which can be rewritten as

$$\int \mathrm{d}^4x \, \varepsilon^{\mu\nu\rho\sigma} \delta\theta^{\alpha\beta}\hat{F}_{\rho\sigma}\left(\hat{X}_{\mu\nu\alpha\beta} + \frac{1}{2}\{\hat{F}_{\alpha\mu} \overset{*}{,} \hat{\delta}_t \hat{B}_{\beta\nu}\} - \frac{1}{2}\{\hat{D}_\alpha \hat{F}_{\mu\nu} \overset{*}{,} \hat{\psi}_\beta\}\right) = 0. \tag{46}$$

Before the final identification, we have to take care to guarantee the covariance of $\hat{X}_{\mu\nu\alpha\beta}$. Remembering the transformation of $\hat{\delta}_t \hat{B}_{\mu\nu}$ in (39), it is not difficult to change the form of the last term in (46), integrating by parts and using a Fierz identity, to arrive at

$$\delta\theta^{\alpha\beta}\hat{X}_{\mu\nu\alpha\beta} = \tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{F}_{[\mu\alpha} \overset{*}{,} \hat{\delta}_t \hat{B}_{\beta\nu]}\} + \tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{F}_{[\mu\alpha} \overset{*}{,} \hat{D}_\beta \hat{\psi}_{\nu]} - \hat{D}_{\nu]}\hat{\psi}_\beta\}. \tag{47}$$

(It must be emphasized that the solution of (47) is still a particular solution of (46). It is always possible to introduce covariant terms satisfying

$$\int \varepsilon^{\mu\nu\rho\sigma} \delta\theta^{\alpha\beta}\hat{F}_{\mu\nu}\hat{X}'_{\rho\sigma\alpha\beta} = 0 \tag{48}$$

with a covariant $\hat{X}'_{\rho\sigma\alpha\beta}$. We will not explore this freedom.)

At this point, we can see that the extended symmetry (18) that we have found at the first explicit order in θ, necessary for the construction of the SW map, is just the first-order integral of (41) with (47).

Finally, we can search for $\delta_\theta \hat{B}_{\mu\nu}$. Applying δ_θ on (37),

$$\delta_\theta \hat{s} \hat{B}_{\mu\nu} = \hat{D}_{[\mu} \, \delta_\theta \overset{\wedge}{\psi}_{\nu]} - \mathrm{i}[\delta_\theta \hat{A}_{[\mu} \overset{*}{,} \overset{\wedge}{\psi}_{\nu]}] + \tfrac{1}{2}\delta\theta^{\alpha\beta}\{\partial_\alpha \hat{A}_{[\mu} \overset{*}{,} \partial_\beta \overset{\wedge}{\psi}_{\nu]}\} + \mathrm{i}[\delta_\theta \hat{c} \overset{*}{,} \hat{B}_{\mu\nu}]$$

$$- \tfrac{1}{2}\delta\theta^{\alpha\beta}\{\partial_\alpha \hat{c} \overset{*}{,} \partial_\beta \hat{B}_{\mu\nu}\} + \mathrm{i}[\hat{c} \overset{*}{,} \delta_\theta \hat{B}_{\mu\nu}] + \delta_\theta \overset{\wedge}{\delta}_t \hat{B}_{\mu\nu}, \tag{49}$$

and then using (30), (31), (34), (41) and (47), we can solve for $\delta_\theta \hat{B}_{\mu\nu}$,

$$\delta_\theta \hat{B}_{\mu\nu} = -\tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{A}_\alpha \overset{*}{,} (\partial_\beta + \hat{D}_\beta)\hat{B}_{\mu\nu}\} - \tfrac{1}{4}\delta\theta^{\alpha\beta}\{\hat{F}_{[\mu\alpha} \overset{*}{,} \hat{B}_{\beta\nu]}\}. \tag{50}$$

We now observe that (22) is the first-order solution of (50), as expected. Then if we substitute the first-order solutions (8) and (22) into (50), we obtain a second-order differential equation in θ, which can be integrated to achieve the second-order expansion of the map $\hat{B}_{\mu\nu}$ (θ, A, B). This can then be iterated order by order. The same reasoning applies to all the other differential equations (30), (31), (33) and (34) for all the fields of the theory. So, this method allows us to solve the problem of the SW map for the noncommutative $\widehat{BF}$ theory order by order in θ. Relation (50) is the last one missing to complete the dependence of the fields of the $\widehat{BF}$ theory on θ.

4. Conclusion

Let us summarize the method up to now. We started from the known BRST transformations of the fields of the NCBF theory in (24). Then, from the SW conditions for these fields, and using the commutation between the commutative BRST operator and the differential operator on the noncommutative parameter (27), we were able to establish the differential equations which determines the dependence of the SW map of the noncommutative fields $\hat{A}_\mu, \hat{c}, \overset{\wedge}{\rho}, \overset{\wedge}{\psi}_\mu$ on θ and on the commutative fields, respectively, equations (30), (31), (33) and (34). Now the explicit form of each map can be obtained by solving these equations order by order in θ (e.g., a first-order solution is just what is obtained by changing $\delta\theta$ by θ and the noncommutative fields by the commutative ones in the right-hand side of (30), (31), (33) and (34)).

The differential equation for the θ dependence of the SW map for $\hat{B}_{\mu\nu}$ is harder to achieve. The origin of this difficulty can be traced back to the lack of the nilpotency of the noncommutative BRST operator on $\hat{B}_{\mu\nu}$, equation (36). To overlap it, and being inspired by our study of the first-order SW map of $\hat{B}_{\mu\nu}$ done in section 2, we admit a possible extension of the BRST transformation of this field on the noncommutative space, equation (37). Now we are able to proceed with our method, by first determining the differential dependence of this BRST extension on θ, equations (41) and (47), and finally obtaining the differential equation for the θ dependence of the SW map for the $\hat{B}_{\mu\nu}$ field itself, equation (50). This solves the question for the existence of the SW map for all the fields of the NCBF theory.

Then, to finish our work, we can now answer the question of what is the behaviour of the action (10) under such map. Its dependence on θ can be obtained by applying δ_θ on (10), and after using (31) and (50), we find

$$\delta_\theta \mathcal{S}_{\widehat{BF}} = \frac{1}{8} \int \mathrm{d}^4x \, \varepsilon^{\mu\nu\rho\sigma} \delta\theta^{\alpha\beta}(\{\hat{F}_{\alpha\mu} \overset{*}{,} \hat{F}_{\beta\nu}\}\hat{B}_{\rho\sigma} - \frac{1}{2}\{\hat{F}_{\alpha\beta} \overset{*}{,} \hat{F}_{\mu\nu}\}\hat{B}_{\rho\sigma} + \hat{F}_{\mu\nu}\{\hat{F}_{\alpha\rho} \overset{*}{,} \hat{B}_{\beta\sigma}\}). \tag{51}$$

By means of a Fierz identity, we can show that the expression on the right-hand side of (51) is null,

$$\delta_\theta \mathcal{S}_{\widehat{BF}} = 0. \tag{52}$$

The final conclusion is that, with the particular Seiberg–Witten transformation determined by (31) and (50), the noncommutative $\widehat{BF}$ action is mapped into its pure commutative version without any nonrenormalizable corrections in θ. The ambiguities in the SW map can generate covariant deformations in θ in the commutative space. But as such deformations only intervene in the interaction sector of the theory, without changing the propagation sector, the topological variables of the BF theory (linking numbers) will remain independent of θ without feeling the presence of these deformations. This is in complete analogy with the case of the noncommutative Chern–Simmons model in 3D [12]. We believe that this renormalizability is a general feature of the commutative theories obtained through the SW map of noncommutative actions with Schwarz-type topological sectors.

Acknowledgments

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